

Perturbative bosonization from two-point correlation functions

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April 30, 2003

Abstract

Here we address the problem of bosonizing massive fermions without making expansions in the fermion masses in both massive QED_2 and QED_3 with N fermion flavors including also a Thirring coupling. We start from two point correlators involving the $U(1)$ fermionic current and the gauge field. From the tensor structure of those correlators we prove that the $U(1)$ current must be identically conserved (topological) in the corresponding bosonized theory both in $D = 2$ and $D = 3$ dimensions. We find an effective generating functional in terms of bosonic fields which reproduces those two point correlators and from that we obtain a map of the Lagrangian density $\bar{\psi}^r (i \not{\partial} - m) \psi^r$ into a bosonic one in both dimensions. This map is nonlocal but it is independent of the electromagnetic and Thirring couplings, at least in the quadratic approximation for the fermionic determinant.

PACS-No.: 11.15.Bt , 11.15.-q

1 Introduction

One among many dreams of the theoretical physicists nowadays is the possibility of extending to higher dimensions ($D > 2$) the bosonization of fermionic models. This can be justified by some good properties of one formulation in contrast with the other one. For instance, strong coupling physics in one model corresponds to weak coupling in the other one. A classic example is the map between the massive Thirring and Sine-Gordon models in $D = 2$. Another interesting aspect of such map is the fact that the usual electromagnetic charge in the fermionic model corresponds to the topological charge of the associated soliton field. Furthermore, we can have a map between a linear theory like massive free fermions and a nonlinear one (Sine-Gordon at $\beta^2 = 4\pi$) on the other side.

In view of these and other interesting properties a lot of work has been devoted to the issue of bosonization [1, 2, 3] (see also [4]). There has also been many attempts to generalize those ideas to higher dimensions [5-22]. For massive fermions in $D = 2$ most of the methods is based on expansions around massless fields which are local conformal theories. In $D = 3$, although the case of massless free fermions can still be mapped into a bosonic theory (see [8]) such theory is nonlocal. Besides, the conformal group is finite in $D = 3$ and not so powerful as in $D = 2$ which makes expansions around the massless case nontrivial. The other possibility is to employ functional methods. Once again the case of massless fermions is easier to deal with since the fermionic determinant can be exactly calculated for $m = 0$. For massive fermions in $D = 2$ a nontrivial Jacobian under chiral transformations plays a key role in deriving the Sine-Gordon model (see [26-30]). In $D = 3$, chiral transformations play no role and although some nonperturbative information is known [21] about the fermionic determinant we are basically left with approximate methods like the one used in this work. On one hand, the method used here is inspired in the approach carried out in [19] which is rather simple and based on two point correlation functions. On the other hand, our bosonization rules departure from those in [19] in the sense that they are independent of the interactions. We extend the approach of [19] by introducing an eletromagnetic interaction and making use of the $1/N$ expansion which allows us to go beyond the lowest-order in the coupling constants. In the next section we introduce the model we are working with and obtain a general expression for the generating funtional of the current and gauge field correlators. The expression is valid for arbitrary dimensions and depends on the vacuum polarization tensor. In sections 3 and 4 we make the calculations explicit in $D = 2$ and $D = 3$ dimensions respectively. We first obtain in the fermionic theory the two point current correlation functions involving also the gauge field and then we write the current in terms of bosonic fields and derive

the corresponding action for such fields which reproduces their correlators. In the final section we draw some conclusions and comment on similar approaches in the literature.

2 Generating Functionals

We start by introducing the notation which will be used in both $D = 2$ and $D = 3$. The generating functional for a generalized QED with Thirring self-interaction is given by

$$Z[J_\mu^r, K_\mu] = \int \mathcal{D}A_\mu \prod_{r=1}^N \mathcal{D}\psi_r \mathcal{D}\bar{\psi}_r \exp \left\{ i \int d^D x \left[-\frac{1}{4} F_{\mu\nu}^2 + \bar{\psi}_r (i \not{\partial} - m - \frac{e}{\sqrt{N}} \not{A}) \psi_r - \frac{g^2}{2N} (\bar{\psi}_r \gamma^\mu \psi_r)^2 + \frac{\lambda}{2} (\partial_\mu A^\mu)^2 + J_\mu^r (\bar{\psi}_r \gamma^\mu \psi_r) + K_\mu A^\mu \right] \right\}, \quad (1)$$

where N is the number of fermion flavors and summation over the repeated flavor index r ($r = 1, 2, \dots, N$) is assumed. It is convenient to introduce an auxiliary vector field B_μ and work with the physically equivalent generating functional:

$$Z[J_\mu^r, K_\mu] = \int \mathcal{D}A_\mu \mathcal{D}B_\mu \prod_{r=1}^N \mathcal{D}\psi_r \mathcal{D}\bar{\psi}_r \exp \left\{ i \int d^D x \left[-\frac{1}{4} F_{\mu\nu}^2 + \frac{1}{2} B_\mu B^\mu + \frac{\lambda}{2} (\partial_\mu A^\mu)^2 + \bar{\psi}_r (i \not{\partial} - m - \frac{e}{\sqrt{N}} \not{A} - \frac{g}{\sqrt{N}} \not{B} + \not{J}^r) \psi_r + K_\mu A^\mu \right] \right\}, \quad (2)$$

Integrating over the fermionic fields we obtain

$$Z[J_\mu^r, K_\mu] = \int \mathcal{D}A_\mu \mathcal{D}B_\mu \exp \left\{ i \int d^D x \left[-\frac{1}{4} F_{\mu\nu}^2 + \frac{1}{2} B_\mu B^\mu + 2K_\mu A^\mu + \frac{\lambda}{2} (\partial_\mu A^\mu)^2 \right] \right\} \times \times \prod_{r=1}^N \det \left[i \not{\partial} - m - \frac{1}{\sqrt{N}} (e \not{A} + g \not{B}) + \not{J}^r \right]. \quad (3)$$

Now we compute the fermionic determinant keeping terms of order $(1/N)^0$ and $(1/N)^{1/2}$. Higher order terms will be neglected. Furthermore, since we are only interested in two point correlators, the terms higher than quadratic in the sources J_μ^r will not be taken into account either. This amounts to the quadratic approximation for the fermionic determinant which gives

$$Z[J_\mu^r, K_\mu] = \int \mathcal{D}A_\mu \mathcal{D}B_\mu \exp \frac{i}{2} \int \frac{d^D k}{(2\pi)^D} \left\{ -\tilde{A}_\mu [\theta^{\mu\nu} k^2 (1 - \lambda) + \lambda k^2 g^{\mu\nu}] \tilde{A}_\nu + \tilde{B}^\mu \tilde{B}_\mu + \sum_{r=1}^N \left(e \frac{\tilde{A}_\mu}{\sqrt{N}} + g \frac{\tilde{B}_\mu}{\sqrt{N}} - \tilde{J}_\mu^r \right) \Pi^{\mu\nu}(k^2) \left(e \frac{\tilde{A}_\nu}{\sqrt{N}} + g \frac{\tilde{B}_\nu}{\sqrt{N}} - \tilde{J}_\nu^r \right) + 2 \tilde{K}_\mu \tilde{A}^\mu \right\} \quad (4)$$

where $\theta^{\mu\nu} = g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}$ and the tilde over the fields represent their Fourier transformations in momentum space. The quantity $\Pi^{\mu\nu}$ is the vacuum polarization tensor:

$$\Pi^{\mu\nu}(k) = i \int \frac{d^D p}{(2\pi)^D} \text{tr} \left[\frac{1}{\not{p} - m + i\epsilon} \gamma^\mu \frac{1}{(\not{p} + \not{k}) - m + i\epsilon} \gamma^\nu \right]. \quad (5)$$

In order to proceed further we have to calculate $\Pi^{\mu\nu}$ which depends on the dimensionality of the space-time.

3 Bosonization from two point correlators in $D = 2$

In this section we restrict ourselves to the $D = 2$ case. Using dimensional regularization we obtain, below the pair creation threshold ($z \equiv \frac{k^2}{4m^2} < 1$), :

$$\Pi_{\mu\nu} = \tilde{\Pi}(k^2) \theta_{\mu\nu} \quad (6)$$

with

$$\tilde{\Pi}(k^2) = \frac{1}{\pi} \left[1 - \frac{1}{[z(1-z)]^{\frac{1}{2}}} \arctan \sqrt{\frac{z}{1-z}} \right]; \quad 0 < z < 1, \quad (7)$$

$$\tilde{\Pi}(k^2) = \frac{1}{\pi} \left[1 - \frac{1}{2} \frac{1}{\sqrt{z(z-1)}} \ln \left(\frac{\sqrt{(1-z)} + \sqrt{-z}}{\sqrt{(1-z)} - \sqrt{-z}} \right) \right]; \quad z < 0. \quad (8)$$

Once the tensor $\Pi^{\mu\nu}$ is calculated one is left with a gaussian integral over the vector fields A_μ and B_μ from which a generating functional quadratic in the sources is derived. Such generating functional furnishes the following two point correlators¹

$$\langle j_\mu^r(k) j_\nu^s(-k) \rangle = -\frac{\tilde{\Pi}^2(e^2 - k^2 g^2)}{ND} \theta^{\mu\nu} + \tilde{\Pi} \theta^{\mu\nu} \delta^{rs}, \quad (9)$$

$$\langle j_\mu^r(k) A_\nu(-k) \rangle = \frac{e \tilde{\Pi}}{\sqrt{ND}} \theta^{\mu\nu}, \quad (10)$$

$$\langle A^\mu(k) A^\nu(-k) \rangle = \left(-\frac{1}{\lambda k^2} + \frac{1 + g^2 \tilde{\Pi}}{D} \right) \theta^{\mu\nu} + \frac{g^{\mu\nu}}{\lambda k^2}, \quad (11)$$

¹More precisely, we should have written explicitly the two point functions in the form $\langle G(k)H(p) \rangle = I(k)\delta^{(2)}(k+p)$ but in this article we will not display the delta function for a matter of convenience. Notice also that when we write $A_{rs} = F + G\delta_{rs}$ it is assumed that F multiplies a $N \times N$ matrix where all entries are equal to one.

where

$$D = [\tilde{\Pi} e^2 - k^2 (1 + g^2 \tilde{\Pi})]. \quad (12)$$

The tensor structure of the above correlation functions are in full agreement with the corresponding Ward identities based on the $U(1)$ symmetry and it will play a key role in our bosonization procedure. At this point it is important to stress that our approach deviates from that of reference [19]. The $1/N$ expansion we have relied upon, which coincides with the quadratic approximation for the fermionic determinant, is not equivalent to the second order weak coupling expansion used in [19].

Now, in order to derive a bosonized expression for the currents $j_\mu^r = \bar{\psi}_r \gamma_\mu \psi_r$, we write down the most general decomposition for a vector in the momentum space:

$$j_\beta^r(k) = \epsilon_{\beta\delta} k^\delta \phi^r(k) + k_\beta \varphi^r(k). \quad (13)$$

Substituting it in (9) and using the identity $\epsilon_{\beta\delta} k^\delta \epsilon_{\alpha\gamma} k^\gamma = k^2 \theta_{\alpha\beta}$ we obtain

$$\begin{aligned} \langle j_\alpha^r(k) j_\beta^s(-k) \rangle &= -k^2 \theta_{\alpha\beta} \langle \phi^r(k) \phi^s(-k) \rangle - k_\alpha k_\beta \langle \varphi^r(k) \varphi^s(-k) \rangle \\ &\quad - \epsilon_{\alpha\delta} k^\delta k_\beta \langle \phi^r(k) \varphi^s(-k) \rangle - \epsilon_{\beta\gamma} k^\gamma k_\alpha \langle \varphi^r(k) \phi^s(-k) \rangle \\ &= -\frac{\tilde{\Pi}^2 (e^2 - k^2 g^2)}{ND} \theta_{\alpha\beta} + \tilde{\Pi} \theta_{\alpha\beta} \delta^{rs}. \end{aligned} \quad (14)$$

From the above it is not difficult to derive

$$\langle \varphi^r(k) \varphi^s(-k) \rangle = 0, \quad (15)$$

$$\langle \phi^r(k) \varphi^s(-k) \rangle = 0, \quad (16)$$

$$\langle \phi^r(k) \phi^s(-k) \rangle = -\frac{1}{k^2} \left[-\frac{\tilde{\Pi}^2 (e^2 - k^2 g^2)}{ND} + \tilde{\Pi} \delta^{rs} \right], \quad (17)$$

from which one can safely set $\varphi = 0$. On the other hand, substituting a general decomposition

$$\langle \phi(k) A_\mu(-k) \rangle = M \epsilon_{\mu\delta} k^\delta + Q k_\mu, \quad (18)$$

in the mixed correlation functions $\langle j_\mu^r(k) A_\nu(-k) \rangle$ given in (10) we conclude that

$$\langle \phi^r(k) A_\mu(-k) \rangle = -\frac{e\tilde{\Pi}}{\sqrt{N}k^2 D} \epsilon_{\mu\delta} k^\delta; \quad (19)$$

Now we are in a position to derive the bosonic Lagrangian density $\mathcal{L}_B(A_\mu, \phi^r)$ which is compatible with the correlation functions (11), (17) and (19). For this purpose we start from the following Ansatz

$$\mathcal{L}_B(A_\mu, \phi^r) = \phi^r R_{rs} \phi^s + 2 S_r \phi^r \epsilon_{\mu\nu} k^\nu A^\mu + A^\mu A^\nu (T_1 \theta_{\mu\nu} + T_2 g_{\mu\nu}), \quad (20)$$

where R_{rs} , S_r , T_1 and T_2 will be determined as follows. We introduce the external sources X^r and K_μ and define the generating functional

$$Z_B[X_r, K_\mu] = \int \prod_{r=1}^N \mathcal{D}\phi^r \mathcal{D}A_\mu e^{i \int d^2x [\mathcal{L}_B(A_\mu, \phi^r) + X_r \phi^r + K_\mu A^\mu]}. \quad (21)$$

Assuming that R_{rs} is a symmetric non-singular matrix we have performed the gaussian integrals in (21) and obtained an explicit formula for $Z_B[X^r, K_\mu]$ from which the two-point correlators can be obtained. By matching these correlators with (11), (17) and (19) we determine the bosonic Lagrangian uniquely :

$$\begin{aligned} \mathcal{L}_B(A_\mu, \phi^r) = & \frac{1}{2} \left[\sum_{r=1}^N \phi^r \frac{k^2}{\tilde{\Pi}} \phi^r + \frac{g^2}{N} k^2 \left(\sum_{r=1}^N \phi^r \right) \left(\sum_{s=1}^N \phi^s \right) \right] \\ & - \frac{A^\alpha A^\beta}{2} \left[(1 - \lambda) k^2 \theta_{\alpha\beta} + \lambda k^2 g_{\alpha\beta} \right] \\ & + \frac{e}{\sqrt{N}} \epsilon_{\mu\nu} A^\mu k^\nu \left(\sum_{r=1}^N \phi^r \right) . \end{aligned} \quad (22)$$

All the steps which lead us to (22) are technically simple and not very elucidating. We should only mention that no expansions on either $1/N$ or any of the coupling constants have been made in those intermediate calculations. Furthermore, notice that if we quantize $\mathcal{L}_B(A_\mu, \phi^r)$ and integrate over the scalar fields ϕ^r in (22) this will lead to a nonlocal effective action for the photon which was studied in [24] where we concluded that, although nonlocal, the theory is free of tachyons. Next, by comparing with the Lagrangian density written in terms of fermionic fields in (1) we have the bosonization formulae for each fermion flavor (no sum over repeated roman indices below) :

$$\bar{\psi}_r \gamma_\mu \psi_r(k) = \epsilon_{\mu\nu} k^\nu \phi^r(k), \quad (23)$$

$$-\bar{\psi}_r(k) (\not{k} + m) \psi_r(k) = \frac{1}{2} \phi^r \frac{k^2}{\tilde{\Pi}} \phi^r. \quad (24)$$

Now some comments are in order. First of all, if for some given flavor we do a $U(1)$ transformation ($\psi_r \rightarrow e^{i\alpha} \psi_r$) in the expectation value $\langle j_\mu^r \rangle$ and use any regularization scheme preserving the $U(1)$ symmetry it will be easy to derive the Ward identity $\langle j_\mu^r \partial^\nu j_\nu^r \rangle = 0$ which implies the tensor structure $\langle j_\mu^r j_\nu^r \rangle \propto \theta_{\mu\nu}$ and consequently we will have (23). So the current is topological due to the $U(1)$ global symmetry and that must hold non-perturbatively. On the other hand, the bosonization rule (24) is only

approximate since the full expression would require, in our approach, the complete knowledge of the fermionic determinant which is only possible for $m \rightarrow 0$. In this case $\tilde{\Pi} \rightarrow 1/\pi$ and we end up with N massless scalar fields topologically coupled to the gauge field. Integration over the gauge field leads to $N - 1$ massless scalar modes and one mode with $m^2 = e^2/(N\pi + g^2)$. Thus, reproducing the particular case of the so called Schwinger-Thirring model for $N = 1$ [25], as well as the Schwinger model result $m^2 = e^2/\pi$ for $g \rightarrow 0$ and $N = 1$. In the opposite limit of large mass ($z \rightarrow 0$) we have $\tilde{\pi} \rightarrow -2z/3$ at leading order. Substituting back in (22) we arrive at a divergent result at $m \rightarrow \infty$ which is in agreement with the $m \rightarrow \infty$ limit of the corresponding Sine-Gordon model. See [32] for a similar comparison in the case of the massive Thirring model without electromagnetic coupling.

4 Bosonization from two-point correlators in $D = 3$

In $D = 3$ dimensions the vacuum polarization tensor (5) calculated by means of dimensional regularization is given by :

$$\Pi^{\mu\nu}(k) = i \Pi_1 E^{\mu\nu} + \Pi_2 k^2 \theta^{\mu\nu}, \quad (25)$$

with $E^{\mu\nu} \equiv \epsilon^{\mu\nu\rho} k_\rho$ and, in the range $0 \leq z < 1$,

$$\Pi_1 = -\frac{1}{8\pi z^{1/2}} \ln \left(\frac{1+z^{1/2}}{1-z^{1/2}} \right); \quad \Pi_2 = \frac{1}{16\pi m z} \left[1 - \left(\frac{1+z}{2z^{1/2}} \right) \ln \left(\frac{1+z^{1/2}}{1-z^{1/2}} \right) \right]. \quad (26)$$

While for $z < 0$ we have

$$\Pi_1 = -\frac{1}{4\pi(-z)^{1/2}} \arctan \sqrt{-z}; \quad \Pi_2 = \frac{1}{16\pi m z} \left[1 - \left(\frac{1+z}{2(-z)^{1/2}} \right) \arctan \sqrt{-z} \right]. \quad (27)$$

In fact the Π_1 amounts to a regularization dependent finite term [6, 7], which was taken equals to zero due the dimensional regularization used.

Substituting (25) in the general expression (4) we can obtain the two-point functions:

$$\begin{aligned} \langle j_\mu^r(k) j_\nu^s(-k) \rangle &= -\frac{1}{N} \left[k^2 \left(\Pi_2 + \frac{P}{\tilde{Q}} \right) \theta_{\mu\nu} + i \Pi_1 \left(1 - \frac{k^2}{\tilde{Q}} \right) E_{\mu\nu} \right] \\ &\quad + \left(k^2 \Pi_2 \theta_{\mu\nu} + i \Pi_1 E_{\mu\nu} \right) \delta_{rs}, \end{aligned} \quad (28)$$

$$\langle j_\mu^r(k) A_\nu(-k) \rangle = \frac{e}{\tilde{Q}\sqrt{N}} [P\theta_{\mu\nu} - i \Pi_1 E_{\mu\nu}], \quad (29)$$

$$\langle A_\mu(k) A_\nu(-k) \rangle = \frac{g_{\mu\nu}}{\lambda k^2} + \left[\frac{e^2 P}{k^2 \tilde{Q}} - \frac{(1+\lambda)}{\lambda k^2} \right] \theta_{\nu\mu} - \frac{ie^2 \Pi_1}{k^2 \tilde{Q}} E_{\mu\nu}, \quad (30)$$

where we found convenient to define

$$P = (e^2 - k^2 g^2)(k^2 \Pi_2^2 - \Pi_1^2) - k^2 \Pi_2, \quad (31)$$

$$\tilde{Q} = k^2 \left[(e^2 - k^2 g^2) \Pi_2 - 1 \right]^2 - (e^2 - k^2 g^2)^2 \Pi_1^2. \quad (32)$$

In Analogy to the $D = 2$ case, we use for the $U(1)$ current a general decomposition in the momentum space :

$$j_\alpha^r(k) = \epsilon_{\alpha\beta\gamma} k^\beta B_r^\gamma(k) + k_\alpha \phi^r(k), \quad (33)$$

from which we get

$$\begin{aligned} \langle j_\alpha^r(k) j_\beta^s(-k) \rangle &= -k_\alpha k_\beta \langle \phi^r(k) \phi^s(-k) \rangle + E_{\alpha\gamma} E_{\delta\beta} \langle B_r^\gamma(k) B_s^\delta(-k) \rangle \\ &+ k_\alpha E_{\beta\delta} \langle \phi^r(k) B_s^\delta(-k) \rangle + k_\beta E_{\alpha\gamma} \langle B_r^\gamma(k) \phi_s(-k) \rangle. \end{aligned} \quad (34)$$

Multiplying the last expression by $k^\alpha k^\beta$ we conclude that $\langle \phi^r(k) \phi^s(-k) \rangle = 0$. Now, multiplying the resulting expression by k^α we have $E_{\alpha\beta} \langle \phi^r(k) B_s^\beta(-k) \rangle = 0$. A similar manipulation was used in the last section to derive (15), (16) and (17). Concluding, we can certainly neglect the scalar fields $\phi^r = 0$ and minimally bosonize the $U(1)$ current in $D = 3$ in terms of a bosonic vector field. The bosonic version of the current is once again of topological nature and identically conserved. As in $D = 2$ case this happens because of the $U(1)$ global symmetry of the fermionic Lagrangian. Taking $\phi^r = 0$ and substituting the decomposition (33) in (28) and (29), after some trivial manipulations, we end up with

$$\langle B_r^\gamma(k) B_s^\delta(-k) \rangle = -C_{rs} g^{\gamma\delta} + \left(C - \frac{H_2}{k^2} \right)_{rs} \theta^{\gamma\delta} - \frac{(H_1)_{rs}}{k^2} E^{\gamma\delta}, \quad (35)$$

$$\langle A^\nu(k) B_s^\mu(-k) \rangle = - \left(\frac{ie\Pi_1}{\sqrt{N}\tilde{Q}} + D \right)_s g^{\nu\mu} - \left(\frac{eP}{\sqrt{N}\tilde{Q}k^2} \right)_s E^{\nu\mu} + D_s \theta^{\nu\mu}, \quad (36)$$

where C_{rs} and D_s are arbitrary and

$$(H_1)_{rs} = i\Pi_1 \delta_{rs} + \frac{i\Pi_1}{N} \left(\frac{k^2}{\tilde{Q}} - 1 \right), \quad (37)$$

$$(H_2)_{rs} = k^2 \Pi_2 \delta_{rs} - \frac{k^2}{N} \left(\Pi_2 + \frac{P}{\tilde{Q}} \right). \quad (38)$$

The arbitrariness of C_{rs} and D_s shows that the bosonization fields B_r^μ must be gauge fields but the corresponding gauge symmetry is independent of the eletromagnetic one. Analogous to the last section, we next derive a Lagrangian density compatible with the two-point correlators (35),(36) and (30). Let us suppose we have a bosonic Lagrangian density $\mathcal{L}_B(A_\mu, B_r^\nu)$ of the form

$$\mathcal{L}_B(A_\mu, B_r^\nu) = B_r^\mu \mathcal{O}_{\mu\nu}^{rs} B_s^\nu + u^s B_s^\mu E_{\mu\nu} A^\nu + A_\mu A_\nu (v_1 \theta_{\mu\nu} + v_2 g_{\mu\nu}), \quad (39)$$

Where $\mathcal{O}_{\mu\nu}^{rs}$ has a general tensor structure in both momentum and flavor space,

$$\mathcal{O}_{\mu\nu}^{rs} = a^{rs} \theta_{\mu\nu} + b^{rs} E_{\mu\nu} + d^{rs} g_{\mu\nu} \quad (40)$$

The quantities $a^{rs}, b^{rs}, d^{rs}, u^s, v_1, v_2$ will be determined by matching the two-point correlators derived from the generating functional :

$$Z_B[Y_\nu^r, K_\mu] = \int \prod_{r=1}^N \mathcal{D}B_r^\mu \mathcal{D}A_\mu e^{i \int d^2x (\mathcal{L}_B(A_\mu, B_r^\nu) + Y_\mu^r B_r^\mu + K_\mu A^\mu)}, \quad (41)$$

with the correlators (35),(36) and (30). Assuming that the matrices a^{rs}, b^{rs}, d^{rs} are symmetric and non-singular, our first step is to perform the gaussian integrals over the N vector fields B_r^ν which furnishes:

$$Z_B[Y_\nu^r, K_\mu] = \int \mathcal{D}A_\mu e^{i \int d^2x [-\frac{1}{4}(Y_r^\mu + u_r A_\beta E^{\beta\mu})(\mathcal{O}^{-1})_{\mu\nu}^{rs}(Y_s^\nu + u_s E^{\nu\beta} A_\beta) + A^\mu A^\nu (v_1 \theta_{\mu\nu} + v_2 g_{\mu\nu}) + K_\mu A^\mu]}. \quad (42)$$

The inverse operator \mathcal{O}^{-1} has the same structure of (40): $(\mathcal{O}^{-1})_{\mu\nu}^{rs} = \tilde{a}^{rs} \theta_{\mu\nu} + \tilde{b}^{rs} E_{\mu\nu} + \tilde{d}^{rs} g_{\mu\nu}$ where the tilde variables are functions of the non-tilde ones. In particular, $\tilde{d}_{rs} = (d^{-1})_{rs}$. Since we have the contractions $E^{\mu\nu} E_{\nu\alpha} = -k^2 \theta_\alpha^\mu$; $E^{\mu\nu} \theta_{\nu\alpha} = E_\alpha^\mu$ it becomes clear from (42), with a little thought, that even after integrating over A_μ the only term in $\log Z_B$ quadratic in the sources Y_r^μ and contracted via the metric tensor is : $-\frac{1}{4} Y_r^\mu Y_s^\nu \tilde{d}^{rs} g_{\mu\nu}$. Therefore the two point functions (35) will require the identification $C^{rs} = (\tilde{d}/2)^{rs} = (d^{-1}/2)^{rs}$.

Given that the matrix C^{rs} is arbitrary, in order to simplify the calculations, we have chosen it proportional to the identity and consequently we reduce the undetermined parameters in our original Ansatz for the bosonic lagrangian since $d^{rs} = d \delta^{rs}$. For analogous reasons, after integrating over A_μ , the crossed terms in the sources Y_r^μ and K^ν will be only contracted by tensors $E_{\mu\nu}$ and $\theta_{\mu\nu}$. Thus, the term proportional to $g^{\nu\mu}$ on the right handed side of the correlation functions (36) must vanish, which fixes $D = -\frac{ie\Pi_1}{\sqrt{NQ}}$. Finally, comparing the other parts of the correlators (35), (36) and (30) with the final expression

for $\log Z_B$ one can determine the quantities a^{rs} , b^{rs} and u^s as functions of “d” which remains unfixed. We spare the reader the lengthy details since they are totally technical and not very illuminating. We only stress that no further simplification hypothesis is used and no expansion is made in neither $1/N$ nor in any of the coupling constants. In those intermediate steps all correlators are treated as if they were exact.

In order to make the final answer for the bosonic Lagrangian density more familiar we have redefined the unfixed parameter $d \equiv -\tilde{\lambda}k^2/2$. In the momentum space we have:

$$\begin{aligned} \mathcal{L}_B(A_\mu, B_r^\alpha) = & -\frac{1}{2} \left[\sum_{r=1}^N \frac{B_r^\alpha (k^2 \Pi_2 \theta_{\alpha\beta} + i \Pi_1 \epsilon_{\alpha\beta\gamma} k^\gamma) B_r^\beta}{k^2 \Pi_2^2 - \Pi_1^2} + \frac{g^2}{N} \left(\sum_{r=1}^N B_r^\alpha \right) k^2 \theta_{\alpha\beta} \left(\sum_{s=1}^N B_s^\beta \right) \right] \\ & - \frac{A^\alpha A^\beta}{2} \left[(1 - \lambda) k^2 \theta_{\alpha\beta} + \lambda k^2 g_{\alpha\beta} \right] + \frac{e}{\sqrt{N}} \epsilon_{\mu\nu\gamma} k^\gamma A^\mu \sum_{r=1}^N B_r^\nu \\ & + \sum_{r=1}^N \frac{B_r^\alpha B_r^\beta}{2} \left[\tilde{\lambda} k^2 \theta_{\alpha\beta} - \tilde{\lambda} k^2 g_{\alpha\beta} \right] . \end{aligned} \quad (43)$$

Notice that λ and $\tilde{\lambda}$ are interpreted as independent gauge fixing parameters such that the generating functional (41) reproduces the correlation functions (30), (35) and (36) with

$$C_{rs} = -\frac{1}{\tilde{\lambda}k^2} \delta_{rs} \quad (44)$$

$$D = -\frac{ie\Pi_1}{\sqrt{N}\tilde{Q}} \quad (45)$$

Integration over the vector fields B_r^α will lead to an effective eletromagnetic theory which was also studied in [24] (see also [26]) where we have analysed its pole structure and concluded that no tachyons appear just like in its two dimensional couterpart (22).

Comparing (43) with the original Lagrangian (1) written in terms of fermionic fields we have the bosonization rules for each fermion flavor ($r = 1, 2, \dots, N$):

$$\bar{\psi}_r \gamma_\mu \psi_r(k) = \epsilon_{\mu\nu\alpha} k^\nu B_r^\alpha(k), \quad (46)$$

$$-\bar{\psi}_r(k) (\not{k} + m) \psi_r(k) = -\frac{1}{2} B_r^\alpha \frac{(k^2 \Pi_2 \theta_{\alpha\beta} + i \Pi_1 E_{\alpha\beta})}{k^2 \Pi_2^2 - \Pi_1^2} B_r^\beta. \quad (47)$$

Similarly to the $D = 2$ case, we have obtained (46) from the fact that for each fermion flavor we have the tensor structure $\langle j_\mu j_\nu \rangle \propto a \theta_{\mu\nu} + b E_{\mu\nu}$, which has been derived by calculating the fermionic determinant up to the quadratic order. However, this tensor structure must hold beyond that approximation (non-perturbatively) as a consequence

of the Ward identity $\langle j_\mu \partial^\nu j_\nu \rangle = 0$ which must be true also in $D = 3$ by using a gauge invariant regularization. On the other hand, the bosonization rule (47) holds as a consequence of the quadratic approximation for the fermionic determinant. Taking $m \rightarrow 0$ (or $z \rightarrow -\infty$) in the expressions (27) we have $\Pi_1 \rightarrow 0$ and $\Pi_2 \rightarrow 1/(16\sqrt{-k^2})$. Plugging back in (46) and (47) we recover the result of [8] with $\beta = 1/4$ and $\theta = 0$ in that reference which deals with massless fermions. Our rules (46) and (47) also agree with the case of massive fermions treated in [15]. We stress however, that in both [8] and [15] only free fermions have been considered while our results have been derived for an interacting theory with Thirring and electromagnetic couplings. This indicates that our bosonization rules are rather universal at the quadratic level. In the opposite limit of heavy fermions ($k^2/4m^2 = z \rightarrow 0$ in (26) we have, at leading order, $\Pi_2 \rightarrow 0$ and $\Pi_1 \rightarrow -1/(4\pi)$ which leaves us, see (47), with a bosonic local and topological Chern-Simons Lagrangian for the field B_r^μ in agreement with the findings of [10] (see also [11] and [12] for higher orders).

5 Conclusions

We have derived bosonization maps for the $U(1)$ currents and the fermion Lagrangian densities for both QED_2 and QED_3 with N fermion flavors and Thirring self-interaction. Both results hold for finite fermion masses and no derivative expansion on $\frac{k}{m}$ is made as in [31]. By turning off the interactions we can reproduce the results of [15] for a particular choice of the regularization parameter in that reference. Our calculations show that the $U(1)$ currents, when written in terms of bosons, must be identically conserved (topological) as a direct consequence of the Ward-identity $\langle j_\mu \partial^\nu j_\nu \rangle = 0$ which is true non-perturbatively if there is no $U(1)$ anomaly. In particular, there is no need for looking at higher point functions or compute the fermionic determinant beyond the quadratic approximation to confirm that. On the other hand, our maps for the Lagrangian densities (24) and (47), which are also independent of the interactions, hold only perturbatively due to our quadratic approximation for the fermionic determinant which amounts to neglect terms of order higher than $(1/N)^{1/2}$ in the fermionic determinant and consider at most two point current correlators to derive the bosonization rules.

Finally, we should mention that the results derived here could have been obtained by a technically more direct way along the lines of [20] (see also [23]) but the approach used here does not use any auxiliary field and clarifies the fundamental role of the two point correlators. Besides, it might be also useful (work in progress) for deriving approximate bosonic maps for other fermion bilinears like the mass term $\bar{\psi}\psi$ in $D = 3$, which apparently could not be done by using the approach of [20].

6 Acknowledgements

This work was partially supported by **CNPq** and **FAPESP**, Brazilian research agencies.

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